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Blochequation for a time varying magnetic field and Sturm Liouville equation

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Abstract:

We consider analytic solution of the Bloch using the concept of a Fundamental matrix, used in a system of differential equations. In a particular case, the magnetization vector satisfies a second-order differential equation, which has the form of the Sturm Liouville equation.

Introduction:

Magnetic Resonance Imaging(MRI) technique is widely usedin medical fieldto study the nuclear magnetisation of the body to be imaged by using different kinds of magnetic fields. Time dependence of nuclear magnetisation $\vec{\boldsymbol{M}}$ under the influence of magnetic field $\vec{B}$ is modelled by the following differential equation proposed by Bloch: [3][9][12]

$$\frac{d\vec{M}}{dt} = \gamma \vec{M} X \vec{B} - \frac{1}{T_2} M_x \vec{\iota} - \frac{1}{T_2} M_y \vec{\jmath} + \frac{1}{T_1}(M_0 - M_z)\vec{k}. \qquad (1)$$

Here$\vec{\iota}, \vec{\jmath}$ and $\vec{k}$ are unit vectors in x, y and z direction in rectangular Cartesians systems.M_x, M_y and M_z are components of $\vec{\boldsymbol{M}}$in Laboratory frame,M_0 is the strength of equilibrium magnetisation ,γ is the gyromagnetic ratioand $T_1$and T_2 are spin-lattice relaxation time and spin spin relaxation time respectively. The equation is valid during relaxation as well as resonance processes.[4].[2] [1]

Bloch equation for time varying magnetic field

Let $B = B_1 f(t)\vec{k}$ be the magnetic field in the z- direction. Then equation (1) becomes[2]

$$\begin{cases} \frac{dM_x}{dt} = -\frac{1}{T_2}M_x + \gamma B_1 f(t) M_y \\ \frac{dM_y}{dt} = -\gamma B_1 f(t) M_x - \frac{1}{T_2}M_y \quad (2) \\ \frac{dM_z}{dt} = \frac{1}{T_1}(M_0 - M_z). \end{cases}$$

Let $\gamma B_1 f(t) = w_1(t)$. Then we get the system

$$\begin{cases} \frac{dM_x}{dt} = -\frac{1}{T_2}M_x + w_1(t) M_y \\ \frac{dM_y}{dt} = -w_1(t) M_x - \frac{1}{T_2}M_y (3) \\ \frac{dM_z}{dt} = \frac{1}{T_1}(M_0 - M_z) . \end{cases}$$

We can see that the first two equations in the system (3) are coupled, but the third one is free from coupling . When the x- and y- components of the magnetic field are zero , the third equation in the system can be solved independently using the technique for variable separable form.[5] Thus we get the solution corresponding to the third equation is given by

$$M_z(t) = e^{-t/T_1} M_z(0) + M_0\left(1 - e^{-t/T_1}\right) \quad (4)$$

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Now we consider first two equations in the system (3),
i.e,

$$\frac{dM_x}{dt} = -\frac{1}{T_2}M_x + w_1(t)M_y$$

and $\frac{dM_y}{dt} = -w_1(t)M_x - \frac{1}{T_2}M_y.$ (5)

This is a system of linear differential equations with variable coefficients.
The coefficient matrix

$$A(t) = \begin{bmatrix} \frac{-1}{T_2} & w_1(t) \\ -w_1(t) & -\frac{1}{T_2} \end{bmatrix} = \begin{bmatrix} \frac{-1}{T_2} & 0 \\ 0 & -\frac{1}{T_2} \end{bmatrix} + \begin{bmatrix} 0 & w_1(t) \\ -w_1(t) & 0 \end{bmatrix},$$

is the sum of a scaled identity and antisymmetric matrices.

We let $\Omega(t) = \int_0^t w_1(\tau)d\tau$, which is a cumulative rotation from 0 to t.
Thus the fundamental matrix $\phi(t) =$

$$\exp\begin{bmatrix} -{}^{t}/_{T_2} & \Omega(t) \\ -\Omega(t) & -{}^{t}/_{T_2} \end{bmatrix}$$

$$= \mathrm{e}^{-t/T_2}\begin{bmatrix} \cos\Omega(t) & \sin\Omega(t) \\ -\sin\Omega(t) & \cos\Omega(t) \end{bmatrix}$$

Thus the general solution of (5) is given by

$$\mathrm{e}^{-t/T_2}\begin{bmatrix} \cos\Omega(t) & \sin\Omega(t) \\ -\sin\Omega(t) & \cos\Omega(t) \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$$

$$= \mathrm{e}^{-t/T_2}\begin{bmatrix} C_1\cos\Omega(t) + C_2\sin\Omega(t) \\ -C_1\sin\Omega(t) + C_2\cos\Omega(t) \end{bmatrix}$$

Applying the initial conditions,i.e, for t=0 ,$M_x = M_x(0)$, $M_y = M_y(0)$,
we get

$$\begin{cases} M_x(t) = e^{-t/T_2}\left(M_x(0)\cos\Omega(t) + M_y(0)\sin\Omega(t)\right) \\ M_y(t) = e^{-t/T_2}\left(-M_x(0)\sin\Omega(t) + M_y(0)\cos\Omega(t)\right) \end{cases} \quad (6)$$

(6)and (4) together give the solution of (3) :

$$\begin{cases} M_x(t) = e^{-t/T_2}\left(M_x(0)\cos\,\Omega(t) + M_y(0)\sin\Omega(t)\right) \\ M_y(t) = e^{-t/T_2}\left(-M_x(0)\sin\Omega(t) + M_y(0)\cos\Omega(t)\right) \\ M_z(t) = e^{-t/T_1}M_z(0) + M_0\left(1 - e^{-t/T_1}\right) \end{cases} \quad (7)$$

Conclusions

1) Under the condition $w_1(t) = ce^{2t/T_2}$, the coupled equation (5) gets transformed to a second order differential equation $\frac{d^2M_x}{dt^2} = \left(\frac{1}{{T_2}^2} - {w_1}^2(t)\right)M_x$ which is a Sturm Liouville equation.
The solution of the system (5) under the above condition can be derived from (6) by substituting $\Omega(t) = \int_0^t ce^{2t/T_2}\,d\tau$

2)Analytic solution for the Bloch equation in the case of constant magnetic field in the z direction $\vec{B} = B_0\vec{k}$,availablealready in the literature, [1][10][11] can be obtained from (7) by substituting $\Omega(t) = \gamma B_0$t. [5]It can also be found by the method of variation of parameters or by using integrating factors. [5][8]

3) In general the relaxation times $T_1$and T_2 are not equal, but when they are equal, the Bloch equation will be highly simplified. [13] [7] Analytic solution in that case can be obtained from (7) by taking $\Omega(t) = \gamma B_0$t and $T_1 = T_2 = T$.

The particular case in which the system reduces to Sturm Liouville equation can opens ways for further studies.

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